Real symmetric random matrices and path counting

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Exact evaluation of $\langle \mathrm{Tr} S^p \rangle$ is here performed for real symmetric matrices S of arbitrary order n, up to some integer p, where the matrix entries are independent identically distributed random variables, with an arbitrary probability distribution. These expectations are polynomials in the moments of the matrix entries; they provide useful information on the spectral density of the ensemble in the large n limit. They also are a straightforward tool to examine a variety of rescalings of the entries in the large n limit.

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I. INTRODUCTION

In past decades the study and applications of random matrix theory in a variety of fields of theoretical physics has witnessed a tremendous expansion. In most cases one is interested in universality properties of the model, which, in the limit of infinite order of the matrices, are independent of the detailed form of probability distribution of the random variables.

Two main classes of models were studied. First, models with independent entries, in which the (independent) matrix entries are independent identically distributed random variables; that is, the joint probability density factorizes. Several matrix ensembles with real or complex entries were studied: from the simplest, where no structure of the matrix is considered, to the sparse matrices (see for example [1–4]); the Laplacian matrices, where the diagonal elements are $s_{j,j} = -\Sigma_{k=1\to n, k\neq j} s_{j,k}$ (see, for example, [5,6]); the band matrices (see, for example, [7–11]); and the block band matrices (see for example [12,13]) second, unitary-invariant models in which the three classic models are real symmetric matrices, complex Hermitian matrices, or matrices where the entries are real quaternions. Here the joint probability density of the random variables is a function invariant under unitary (real or complex) transformations. The most studied models have the joint probability density in the form of a Boltzmann factor, which, for real symmetric matrices, has the form

$$p(s_{1,1}, s_{1,2}, \ldots, s_{n,n}) = e^{-\mathrm{Tr}\ V(S)}, \tag{1.1}$$

where $V(x)$ is a polynomial in the variable x. The important case where $V(x) = x^2$ describes the Gaussian orthogonal ensemble, which is a model also of independent entries. Further important matrix ensembles, such as the Wishart matrices, were studied with probability density that often do not belong to the above two classes.

Techniques to study the two classes of models are quite different. In the second class, by using the unitarity invariance, the model is rewritten in terms of the eigenvalues of the random matrix. Powerful analytic tools are available, such as the technique of orthogonal polynomials. It was found that although the limiting spectral density is sensitive to the details of $V(x)$, several important features related to the local statistics of eigenvalues exhibit universality properties.

In the models of the first class, few analytical tools are available and often the limiting spectral density is obtained by numerical evaluation from samples of large matrices. The most famous result is the universality property of the spectral density proved by Wigner [7]: with suitable assumptions (which will be reviewed later) the limiting spectral density is the semicircle, independent from the probability distribution of the entries of the random matrix.

In this paper few simple ensembles of real symmetric matrices of the first class will be analyzed: the $\{S\}$ ensemble of real symmetric matrices, the closely related $\{S_0\}$ ensemble (that will be defined in Sec. II), the random sparse matrices, and the bidiagonal matrices (the simplest type of band matrices).

Let $\{S\}$ be the ensemble of real symmetric matrices of order n. Any matrix $S \in \{S\}$ is defined by $n(n+1)/2$ real independent matrix elements s_{jk}. The joint probability density factorizes

$$p(s_{1,1}, s_{1,2}, \ldots, s_{n,n}) = \prod_{i\leq j} p(s_{i,j}). \tag{1.2}$$

Let x denote any of the independent entries of the random matrix, $p(x)$ its probability density, and $\langle x^k \rangle$ its moments

$$E[x^k] = \langle x^k \rangle = \int dx x^k p(x). \tag{1.3}$$

We shall only assume that the moments of any degree exist, and in Sec. II the expectations $E[\mathrm{Tr} S^p]$ are exactly evaluated as polynomials in the moments $\langle x^k \rangle$ for any value of the matrix size n and for p up to 7 for the ensemble $\{S\}$, up to $p=8$ for the related ensemble $\{S_0\}$, any p for the bidiagonal matrices. After a proper n-dependent rescaling, the evaluation of $E[\mathrm{Tr} S^p]$, give the moments of the limiting spectral density, if $E(x)=0$. Symbolic manipulation software is efficient in sorting out words of given type and counting them. Then the exact evaluation in Eqs. (2.10) and (2.12), may be obtained by such software by using matrices with symbolic entries of several orders, say $n=4-9$, enough to determine the polynomials in the n variable. However, computer

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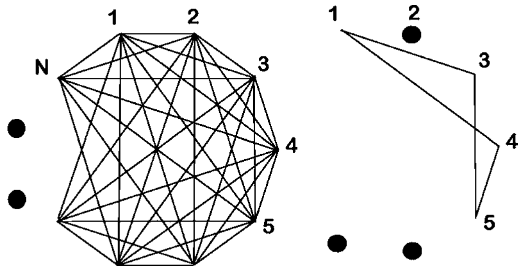

FIG. 1. The left side shows the complete graph with N vertices, the right side shows a four-step path on it.

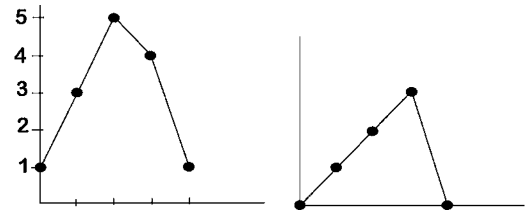

FIG. 2. In both graphs, time is the horizontal axis. The left side shows the history of a path on the graph by exhibiting the visited sites. The right side is the graph of a reduced path corresponding to the graph on the left side.

memory limits a brute force approach when the number of words is a few hundred thousand. In Sec. II a simple description is given in terms of (non-Markov) paths, which encode all the relevant information. We found combining such manual counting with the symbolic software counting useful.

The evaluation of the moments of the spectral density by counting classes of paths on the graph associated to the matrix has long history (see, for example, [3,4,7,14–16]). One may see [17,18] for different methods. Usually, the aim was to evaluate the number of classes of paths for large n and large p, with $n \gg p$.

The finite p exact evaluation provided in Sec. II will probably be useful for two purposes: (i) in several models of sparse matrices or Laplacian matrices the limiting spectral density is determined numerically by extrapolating evaluations for several values of n. In such cases, the knowledge of the first few moments of the spectral density, provided here, may add useful information. (ii) the results (2.10) and (2.12) allow a transparent analysis of the n-dependent rescaling preliminary to the $n \to \infty$ limit. Then it becomes evident why universality of the limiting spectral density is obtained in the case of Wigner scaling and not in the case of sparse matrices scaling. This will be shown in Sec. III, and the possibility of different scalings will be considered.

II. ENUMERATION OF PATHS

It is well known that to any square matrix $A=\{a_{ij}\}$, $n \times n$ one may associate a graph with n vertices and a directed edge from vertex i to vertex j corresponding to the matrix element a_{ij}. One can the visualize the evaluation of matrix elements of powers of A as sum of paths on the graph. For example $a_{1,4}a_{4,2}a_{2,3}$ corresponds to the three-step path on the graph from vertex 1 to vertex 3 going first to vertex 4 then to vertex 2. Similarly, the matrix element $(A^3)_{1,3} = \Sigma_{j,k=1,\ldots,n} a_{1,j}a_{j,k}a_{k,3}$ corresponds to the sum of n^2 paths on the graph from vertex 1 to vertex 3, visiting all possible intermediate vertices j, k.

We begin by considering the ensemble S_0 of real symmetric matrices of order n, with vanishing entries on the diagonal, $s_{j,j}=0$ for $j=1,2,\ldots,n$. Because $s_{i,j}=s_{j,i}$ instead of drawing two opposite directed edges between the vertices i and j, one draws just one nondirected edge. The graph associated with the generic matrix S_0 is then the complete graph with n vertices, shown on the left side of Fig. 1

The right side of Fig. 1 shows the four-step path on the graph visiting the sequence of vertices {1, 3, 5, 4, 1}. It corresponds to the product $s_{1,3}s_{3,5}s_{5,4}s_{4,1}$, which contributes to Tr S_0^4. The same path is depicted on the left side of Fig. 2, where the horizontal axes represents time or number of steps. This type of lattice path is usually called an excursion or a bridge, respectively, if the path is constrained in the upper half plane or it is not constrained (see, for, instance, [19]).

The $n(n-1)/2$ independent entries of a matrix of the ensemble $\{S_0\}$ are independent identically distributed random variables. Let us denote with $\langle v^k \rangle$ the k-moment of the probability density. Then

$$E[s_{1,3}s_{3,5}s_{5,4}s_{4,1}] = \langle v \rangle^4. \tag{2.1}$$

Of course the same contribution is obtained from any other four-step path from vertex one to itself, such that the four entries $s_{i,j}$ are all different. Because of the symmetry of the complete graph, the same is true for any path beginning and ending at a vertex different from one. One is led to a more efficient representation of paths that keeps only the relevant information.

The paths on the left side of Fig. 2 are redrawn on the right side of Fig. 2, again with time on the horizontal axis. It reads, start from an arbitrary vertex, next a different vertex, next a vertex different from the two previous ones, next a vertex different from the three previous ones, finally the first vertex. It is trivial to evaluate the number of paths of this kind and obtain the contribution

$$n(n-1)(n-2)(n-3)\langle v \rangle^4.$$

We shall call the graphs, such as the right side of Fig. 2, reduced paths, which correspond to a multitude of paths where the visited sites are listed, such as on the left side of Fig. 2. To evaluate $\langle \mathrm{Tr} S_0^4 \rangle$, only four reduced paths, shown in Fig. 3, are needed. Their contribution is

$$\langle \mathrm{Tr}\ S_0^4 \rangle = n(n-1)\langle v^4 \rangle + 2n(n-1)(n-2)\langle v^2 \rangle^2 + n(n-1)(n-2)(n-3)\langle v \rangle^4. \tag{2.2}$$

From now on we shall only refer to these type of paths and shall call closed paths those where the final vertex is the same as the beginning vertex. Open paths have a final vertex different from the beginning vertex. Only closed paths contribute to the evaluation of Tr S_0^p, but open paths are equally important because all closed paths with p steps are obtained by considering all open paths with $p-1$ steps and adding one further edge connecting the last vertex to the first vertex (closing the path).

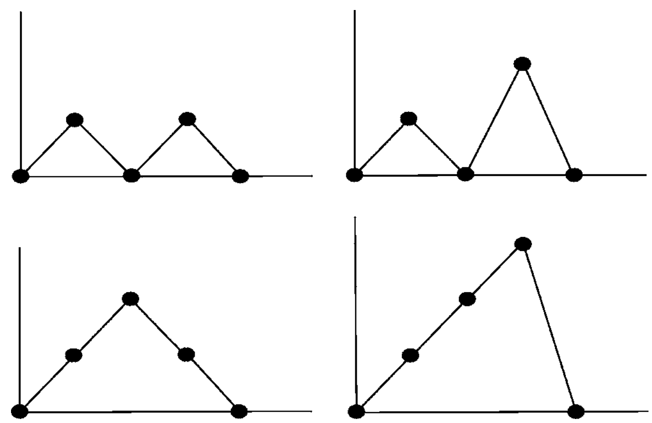

FIG. 3. The four reduced paths, which yield $\langle \mathrm{Tr} S_0^4 \rangle$.

Let $S_c(n,q)$ denote the number of closed paths with n steps such that the maximal altitude reached during the path is q. It means that the path visited a total of q distinct vertices of the complete graph, including the initial vertex. Similarly $S_o(n,q)$ will denote the number of open paths with n steps such that the maximal altitude reached during the path is q. The previous remark translates into

$$S_c(n,q) = S_o(n-1,q). \tag{2.3}$$

One may evaluate the total number of paths, that is, the open plus closed paths for any number of steps.

Consider a path with $n-1$ steps and denote by y_k, $k=0,1,\cdots,n-1$, the altitude of the path at time k. Let $q=\text{maximum}\{y_k\}$. One further step may be added to the path by connecting y_{n-1} with y_n, where y_n is any of the $q+1$ integers $0,1,2,\ldots,q+1$, excluding $y_n=y_{n-1}$. Of course, there is no path where $n<q$.

Let $S(n,q)=S_c(n,q)+S_o(n,q)$ be the total number of paths with n steps and maximal altitude q. The previous construction of paths of n steps from paths with $n-1$ steps translates into

$$S(n,q) = qS(n-1,q) + S(n-1,q-1),$$

$$S(n,q) = 0 \quad \text{if} \quad n<q; \quad S(n,1) = 1. \tag{2.4}$$

The positive integers $S(n,q)$ are known as Stirling numbers of the second kind. Their sum over the values of q are known as Bell numbers $B(n)$ (see, for example, [20,21]). Useful formulas are

$$S(n,q) = \frac{1}{q!}\sum_{r=1}^{q} (-1)^{q-r}\binom{q}{r} r^q,$$

$$B(n) = \sum_{q=1}^{n} S(n,q),$$

$$B(n+1) = \sum_{r=0}^{n} \binom{n}{r} B(r),$$

$$\exp(e^x - 1) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n B(n). \tag{2.5}$$

For example, the total number of paths with four steps is $B(4)=15$ and $S(4,1)=1$, $S(4,2)=7$, $S(4,3)=6$, $S(4,4)=1$. The first few Bell numbers are 1, 2, 5, 15, 52, 203, 877, 4140,… .

Because of Eq. (2.3), $S_c(n)=S_o(n-1)=B(n-1)-S_c(n-1)$, then the first few numbers of closed paths are

$$S_c(2)=1, \quad S_c(3)=1, \quad S_c(4)=4, \quad S_c(5)=11, \quad S_c(6)=41,$$

$$S_c(7)=162, \quad S_c(8)=715, \cdots . \tag{2.6}$$

By substituting all the nonvanishing entries of the matrix S_0 with one, a circulant matrix is obtained where it is easy to evaluate

$$\mathrm{Tr}\ S_0^{2p} = (n-1)[(n-1)^{2p-1}+1],$$

$$\mathrm{Tr}\ S_0^{2p+1} = (n-1)[(n-1)^{2p}-1]. \tag{2.7}$$

One obtains a new and easy determination of the number of closed paths $S_c(n,q)$ by comparing the above numbers with the contribution of the graphs. For instance, in the case of paths with eight steps, by solving the equation

$$(n-1)[(n-1)^7+1] = \sum_{q=1}^{8} n(n-1)(n-2)\cdots(n-q)S_c(8,q), \tag{2.8}$$

we obtain

$$S_c(8,1)=1, \quad S_c(8,2)=42, \quad S_c(8,3)=231,$$

$$S_c(8,4)=294, \quad S_c(8,5)=126, \quad S_c(8,6)=20,$$

$$S_c(8,7)=1, \quad \sum_{q=1}^{8} S_c(8,q) = 715. \tag{2.9}$$

Finally, one draws the graphs and reads their contributions. It is a time-consuming activity, helped by symbolic manipulation software. We report here the evaluations up to $E[\mathrm{Tr} S_0^8]$, which results from 715 graphs

$$\frac{1}{n}\langle \mathrm{Tr}\ S_0 \rangle = 0,$$

$$\frac{1}{n}\langle \mathrm{Tr}\ S_0^2 \rangle = (n-1)\langle v^2 \rangle,$$

$$\frac{1}{n}\langle \mathrm{Tr}\ S_0^3 \rangle = (n-1)(n-2)\langle v \rangle^3,$$

$$\frac{1}{n}\langle \mathrm{Tr}\ S_0^4 \rangle = (n-1)\langle v^4 \rangle + 2(n-1)(n-2)\langle v^2 \rangle^2 + (n-1)(n-2) \times (n-3)\langle v \rangle^4,$$

$$\frac{1}{n}\langle \mathrm{Tr}\ S_0^5\rangle = 5(n-1)(n-2)\langle v^3\rangle\langle v\rangle^2 + 5(n-1)(n-2)(n-3)\times\langle v^2\rangle\langle v\rangle^3 + (n-1)(n-2)(n-3)(n-4)\langle v\rangle^5,$$

$$\frac{1}{n}\langle \mathrm{Tr}\ S_0^6\rangle = (n-1)\langle v^6\rangle + 6(n-1)(n-2)\langle v^4\rangle\langle v^2\rangle + (n-1)(n-2)(5n-11)\langle v^2\rangle^3 + 6(n-1)(n-2)(n-3)\langle v^3\rangle\times\langle v\rangle^3 + 3(n-1)(n-2)(n-3)(2n-5)\langle v^2\rangle\langle v\rangle^4 + (n-1)(n-2)^2(n-3)(n-4)\langle v\rangle^6,$$

$$\frac{1}{n}\langle \mathrm{Tr} S_0^7\rangle = 7(n-1)(n-2)\langle v^5\rangle\langle v\rangle^2 + 14(n-1)(n-2)\langle v^3\rangle^2\langle v\rangle + 7(n-1)(n-2)(n-3)\langle v^4\rangle\langle v\rangle^3 + 35(n-1)(n-2)\times(n-3)\langle v^3\rangle\langle v^2\rangle\langle v\rangle^2 + 7(n-1)(n-2)(n-3)(3n-8)\langle v^2\rangle^2\langle v\rangle^3 + 7(n-1)(n-2)(n-3)(n-4)\langle v^3\rangle\times\langle v\rangle^4 + 7(n-1)(n-2)^2(n-3)(n-4)\langle v^2\rangle\langle v\rangle^5 + (n-1)(n-2)(n-3)(n-4)(n^2-4n+2)\langle v\rangle^7,$$

$$\frac{1}{n}\langle \mathrm{Tr} S_0^8\rangle = (n-1)\langle v^8\rangle + (n-1)(n-2)[8\langle v^6\rangle\langle v^2\rangle + 6\langle v^4\rangle^2] + (n-1)(n-2)^2 28\langle v^4\rangle\langle v^2\rangle^2 + (n-1)(n-2)(n-3)\times[8\langle v^5\rangle\langle v\rangle^3 + 20\langle v^3\rangle^2\langle v\rangle^2] + (n-1)(n-2)^2(n-3)48\langle v^3\rangle\langle v^2\rangle\langle v\rangle^3 + 4(n-1)(n-2)(n-3)(2n-3)\langle v^4\rangle\langle v\rangle^4 + (n-1)(n-2)(n-3)(14n-19)\times\langle v^2\rangle^4 + 2(n-1)(n-2)(n-3)(14n^2-66n+51)\times\langle v^2\rangle^2\langle v\rangle^4 + 8n(n-1)(n-2)(n-3)(n-4)\langle v^3\rangle\times\langle v\rangle^5 + 8(n-1)(n-2)(n-3)(n-4)(n^2-3n-2)\times\langle v^2\rangle\langle v\rangle^6 + (n-1)(n-2)(n-3)(n-4)(n-5)(n^2-n-4)\langle v\rangle^8. \tag{2.10}$$

Bauer and Golinelli, in their analysis of the incidence matrix for random graphs [1], evaluated traces of powers of a random matrix by evaluating normalized k-plets. These are sequences of k positive integers $(v_1, v_2, \dots, v_k)$ such that (i) $v_1 \neq v_2$, $v_2 \neq v_3, \dots, v_{k-1} \neq v_1$ and (ii) if $v_\beta > 1$ is in a normalized k-plets, then also $v_{\beta'} = v_\beta - 1$ is in the same normalized k-plet and $\beta' < \beta$. The normalized k-plets are in one-to-one correspondence with the reduced closed paths described in this paper: the sequence of the k integers of a normalized k-plet is the sequence of the y coordinates $\{y_0 = 1, y_2, \dots, y_{k-1}\}$, of a reduced closed path with k steps. For example, the four reduced paths in Fig. 3 correspond to the four normalized 4-plets {1,2,1,2}, {1,2,1,3}, {1,2,3,1}, {1,2,3,4}. We prefer the graph representation because it is easier to read the repeated steps and allows several generalizations for random matrices with diagonal elements not equal to zero, which may have the same probability distribution of the other entries, as in Sec. II A or a different probability distribution, as it occurs in Laplacian matrices, not described in this paper.

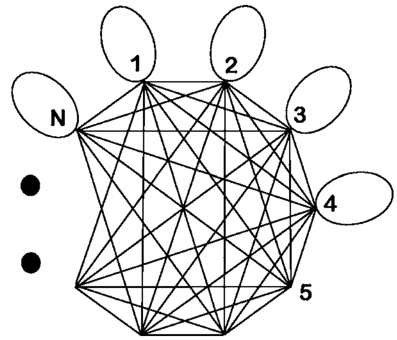

FIG. 4. The complete graph with N vertices corresponding to the symmetric matrix S, with nonzero entries on the diagonal.

A. From the ensemble $\{S_0\}$ to the ensemble $\{S\}$

The graph corresponding to a matrix of the ensemble $\{S\}$, of order n, is the complete graph with n vertices, with a loop added to each vertex. It is depicted in Fig. 4.

The evaluation of $E[\mathrm{Tr} S^p]$ may be performed by considering the closed graphs with p steps where any number of horizontal steps are now allowed at any vertex. A simple example will show how the new reduced paths may be obtained from the reduced paths contributing to the evaluation of $E[\mathrm{Tr} S_0^k]$, $k=0,1,\dots,p$ (where horizontal steps are missing) by adding the $p-k$ horizontal steps in all possible ways.

For example to evaluate $E[\mathrm{Tr} S^4]$, we consider the two paths in Fig. 5, respectively, of two and three steps.

Let us first consider the graph of three steps. It has four vertices, and we obtain four closed graphs contributing to $E[\mathrm{Tr} S^4]$ by adding one horizontal step in four possible ways, as it is shown in Fig. 6. Each of the four graphs corresponds to the contribution $n(n-1)(n-2)\langle v\rangle^4$.

Next one adds two horizontal steps to the graph of two steps in Fig. 5, in six possible ways, thus obtaing the six graphs shown in Fig. 7. In the first four of them, the two horizontal steps are at the same vertex and their contribution is $4n(n-1)\langle v^2\rangle^2$. The last two graphs have the two horizontal steps in different vertices, then their contribution is $2n(n-1)\langle v^2\rangle\langle v\rangle^2$.

Finally, one adds the contribution of one graph with four horizontal steps; that is, $n\langle v^4\rangle$. By collecting the new terms, one obtains

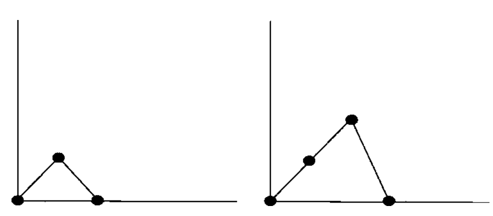

FIG. 5. Two reduced paths of two and three steps.

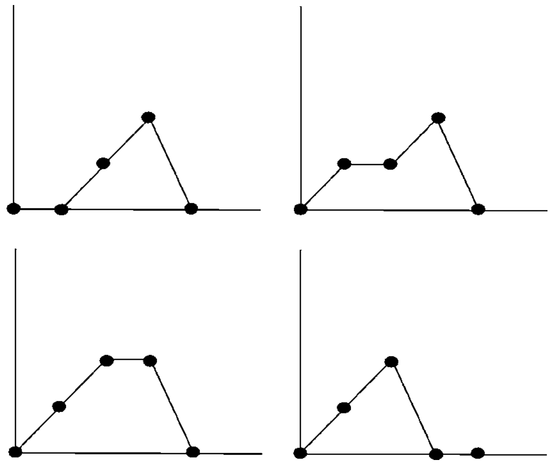

FIG. 6. One horizontal step has been added in four possible ways to the three-step graph of Fig. 5.

$$E[\mathrm{Tr}S^4] = n\langle v^4\rangle + 4n(n-1)\langle v^2\rangle^2 + 2n(n-1)\langle v^2\rangle\langle v\rangle^2 + 4n(n-1)(n-2)\langle v\rangle^4 + E[\mathrm{Tr}S_0^4] = n^2\langle v^4\rangle + 2n(n-1)\langle v^2\rangle \times\langle v\rangle^2 + 2n^2(n-1)\langle v^2\rangle^2 + n(n^2-1)(n-2)\langle v\rangle^4. \quad (2.11)$$

As a trivial check, by replacing all the moments $\langle v^k\rangle \to 1$, one finds $E[\mathrm{Tr}S^4]=n^4$.

One may then evaluate

$$\frac{1}{n}\langle \mathrm{Tr}S\rangle = \langle v\rangle,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^2\rangle = n\langle v^2\rangle,$$

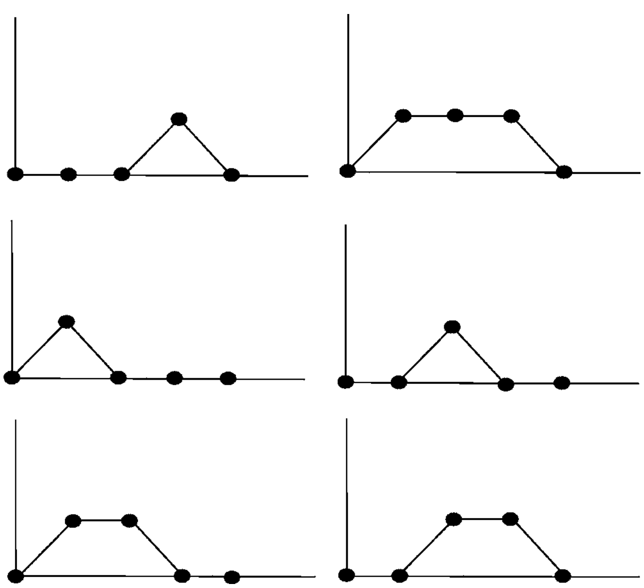

FIG. 7. The six graphs are generated by adding two horizontal steps to the two-step graph of Fig. 5.

$$\frac{1}{n}\langle \mathrm{Tr}S^3\rangle = \langle v^3\rangle + 3(n-1)\langle v^2\rangle\langle v\rangle + (n-1)(n-2)\langle v\rangle^3,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^4\rangle = n\langle v^4\rangle + 2(n-1)\langle v^2\rangle\langle v\rangle^2 + 2n(n-1)\langle v^2\rangle^2 + (n^2-1) \times(n-2)\langle v\rangle^4,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^5\rangle = \langle v^5\rangle + 5(n-1)\langle v^4\rangle\langle v\rangle + 5(n-1)\langle v^3\rangle\langle v^2\rangle + 5(n-1) \times(2n-3)\langle v^2\rangle^2\langle v\rangle + 5(n-1)(n-2)\langle v^3\rangle\langle v\rangle^2 + 5(n-1)(n-2)^2\langle v^2\rangle\langle v\rangle^3 + (n-1)(n-2)(n^2-2n+2) \times\langle v\rangle^5,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^6\rangle = n\langle v^6\rangle + 6(n-1)\langle v^4\rangle\langle v\rangle^2 + 6(n-1)\langle v^3\rangle\langle v^2\rangle\langle v\rangle + 3(n-1)(2n+1)\langle v^4\rangle\langle v^2\rangle + 15(n-1)(n-2) \times\langle v^2\rangle^2\langle v\rangle^2 + (n-1)(5n^2-6n-5)\langle v^2\rangle^3 + 6(n-1) \times(n-2)(n+3)\langle v^3\rangle\langle v\rangle^3 + 3(n-1)(n-2)(2n^2+n-17)\langle v^2\rangle\langle v\rangle^4 + (n-1)(n-2)(n^3-3n^2-7n+23) \times\langle v\rangle^6,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^7\rangle = \langle v^7\rangle + 7(n-1)\langle v^6\rangle\langle v\rangle + 14(n-1)\langle v^4\rangle\langle v^3\rangle + 7(n-1)\langle v^5\rangle\langle v^2\rangle + 14(n-1)(3n-4)\langle v^4\rangle\langle v^2\rangle\langle v\rangle + 7(n-1)(n-2)\langle v^5\rangle\langle v\rangle^2 + 7(n-1)(3n-5)\langle v^3\rangle\langle v^2\rangle^2 + 14(n-1)(n-2)\langle v^3\rangle^2\langle v\rangle + 7(n-1)(n-2)^2\langle v^4\rangle \times\langle v\rangle^3 + 7(n-1)(n-2)(5n-8)\langle v^3\rangle\langle v^2\rangle\langle v\rangle^2 + 35(n-1)^2(n-2)\langle v^2\rangle^3\langle v\rangle + 7(n-1)(n-2)(n^2+1)\langle v^3\rangle\langle v\rangle^4 + 7(n-1)^2(n-2)(3n-8)\langle v^2\rangle^2\langle v\rangle^3 + 7(n-1)(n-2)(n^3-2n^2-2n-2)\langle v^2\rangle\langle v\rangle^5 + (n-1)(n-2)(n-3)(n^3-n^2-10n-1)\langle v\rangle^7 \quad (2.12)$$

B. Bidiagonal symmetric matrices

In a lattice in one dimension with n sites and periodic boundary conditions, each site has two next neighbors, and the lattice has n edges. To each edge $(k,k+1)$ we associate the random variable $a_{k,k+1}=a_{k+1,k}$.

We consider the real symmetric bidiagonal random matrix A

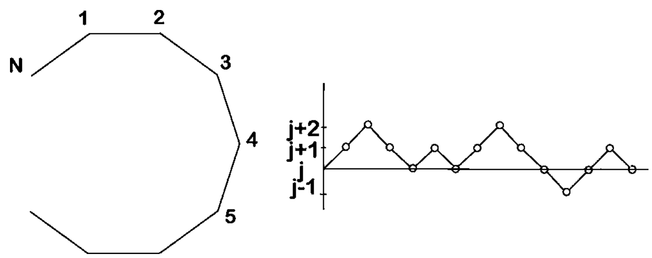

FIG. 8. The left side represents the graph corresponding to the bidiagonal matrix in Eq. (2.13). The right side represents a 14-step path on the graph beginning and ending at site j. Time is the horizontal axis.

$$A = \begin{pmatrix} 0 & a_{1,2} & 0 & 0 & \cdots & 0 & a_{1,n} \\ a_{1,2} & 0 & a_{2,3} & 0 & \cdots & 0 & 0 \\ 0 & a_{2,3} & 0 & a_{3,4} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_{1,n-2} & 0 & a_{n-1,n} \\ a_{1,n} & 0 & 0 & 0 & \cdots & a_{n-1,n} & 0 \end{pmatrix}. \tag{2.13}$$

A walk on the lattice with p steps may return to the initial site only if p is even. The number of such paths[1] is $\binom{p}{p/2}$. This is the number of terms contributing to $(1/n)\langle \mathrm{Tr} A^p \rangle$.

The left side of Fig. 8 shows the periodic one-dimensional lattice with n vertices, associated with the matrix A in Eq. (2.13). The right side of Fig. 8 shows a path from site j returning to site j after 14 steps, then contributing to Tr A^{14}.

Any path may be seen as composed of paths staying (strictly or weakly) in the positive or negative half plane. Paths that do not enter the negative half plane (these are the weakly positive paths), such as the one depicted in the left side of Fig. 9, are known as Dyck paths. Dyck paths have a one-to-one correspondence with rooted trees, as indicated in the right side of Fig. 9. It is the possible to enumerate the number of steps (up or down) between levels k and $k+1$ for the class of Dyck paths of any length. Next by composing weakly positive and weakly negative Dyck paths, one obtains enumeration of steps for the generic paths (see, for instance, [22]).

One easily evaluates

$$\frac{1}{n}\langle \mathrm{Tr} A^{2p+1} \rangle = 0, \quad \frac{1}{n}\langle \mathrm{Tr} A^{0} \rangle = 1,$$

$$\frac{1}{n}\langle \mathrm{Tr} A^{2} \rangle = 2\langle v^2 \rangle,$$

$$\frac{1}{n}\langle \mathrm{Tr} A^{4} \rangle = 2\langle v^4 \rangle + 4\langle v^2 \rangle^2,$$

[1] Because of the periodic boundary conditions, there exist walks on the lattice which return to the initial site from the opposite side of the first step, if $p \geqslant n$. Since we are interested in the limit $n \to \infty$ with fixed p, those walks are neglected here.

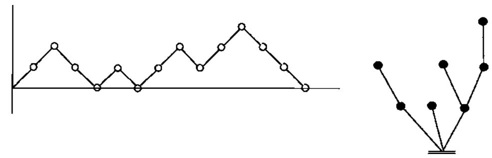

FIG. 9. One-to-one correspondence between Dyck paths and rooted trees.

$$\frac{1}{n}\langle \mathrm{Tr} A^{6} \rangle = 2\langle v^6 \rangle + 12\langle v^4 \rangle\langle v^2 \rangle + 6\langle v^2 \rangle^3,$$

$$\frac{1}{n}\langle \mathrm{Tr} A^{8} \rangle = 2\langle v^8 \rangle + 16\langle v^6 \rangle\langle v^2 \rangle + 12\langle v^4 \rangle^2 + 32\langle v^4 \rangle\langle v^2 \rangle^2 + 8\langle v^2 \rangle^4,$$

$$\frac{1}{n}\langle \mathrm{Tr} A^{10} \rangle = 2\langle v^{10} \rangle + \ + 20\langle v^8 \rangle\langle v^2 \rangle + 40\langle v^6 \rangle\langle v^4 \rangle + 50\langle v^6 \rangle\langle v^2 \rangle^2 + 70\langle v^4 \rangle^2\langle v^2 \rangle + 60\langle v^4 \rangle\langle v^2 \rangle^3 + 10\langle v^2 \rangle^5. \tag{2.14}$$

Figure 9 shows the one-to-one correspondence between Dyck paths and rooted trees: given the tree, one may follow its external contour, starting at the left side of the root. As one moves up along the first branch, one generates two up steps of the Dyck path, then moving down along the same branch generates the two steps down. The next branch is shorter, corresponding to one step up and one step down for the Dyck path. Finally, after touring the next branch, one reaches the right side of the root.

III. SCALINGS AND UNIVERSALITY

To study the existence of a limiting spectral density in the $n \to \infty$ limit, some n-dependent scaling for the entries of the matrices is needed. One may recall that the existence of finite large-n limit of a finite number of rescaled traces of powers of the random matrices is neither necessary nor a sufficient condition for the existence of a limiting spectral density. Indeed, in the case of Gaussian matrices when $\langle v \rangle$ is not equal to zero and Wigner scaling (which will be recalled here) the large-n spectral density of the ensemble contains the classical semicircle separated by a small peak, which disappears at $n = \infty$, yet produces the large-n behavior of the traces.[2] A more complex distribution with possibly several peaks occurs with the sparse matrices [1,6,14].

Conversely, a smooth limiting spectral density for the matrix ensemble may exist, with unlimited support, and decreasing as an inverse power for large eigenvalues. Then only a finite number of moments of the spectral density will exist.

Then the aim of this section is not to derive the limiting spectral density from Eqs. (2.10) and (2.12) for $\langle \mathrm{Tr}\, S^p \rangle$ and $\langle \mathrm{Tr}\, S_0^p \rangle$ after proper rescaling, but merely to review the ef-

[2] We suppose this is well known to experts, but we have not found a proper reference, and then offer the short discussion in the Appendix, by using the addition law for fixed plus random ensembles.

fects of two well-known scalings and to indicate the existence of intermediate scalings that appear to yield the same results of the Wigner scaling for every finite p.

A. Wigner scaling

The Wigner scaling consists of two assumptions: (i) the i.i.d. entries have vanishing expectation value, that is $E(s_{i,j})=\langle v\rangle=0$ for any $i,j=1,2,\ldots,n$; and (ii) the i.i.d. entries are written

$$s_{i,j}=\frac{\xi_{i,j}}{\sqrt{n}},\quad \xi_{i,j}\text{ independent on } n$$

$$\text{then } \mathrm{E}[(s_{i,j})^k]=\langle v^k\rangle=\frac{1}{n^{k/2}}c_k,\quad c_k\text{ independent on } n. \tag{3.1}$$

By inserting these assumptions into Eqs. (2.10) and (2.12) and by keeping only the leading-order contribution in the limit $n\to\infty$, both equations collapse into the very simple result

$$\frac{1}{n}\langle \mathrm{Tr}S^2\rangle=\langle\xi^2\rangle,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^4\rangle=2\langle\xi^2\rangle^2+O\left(\frac{1}{n}\right),$$

$$\frac{1}{n}\langle \mathrm{Tr}S^6\rangle=5\langle\xi^2\rangle^3+O\left(\frac{1}{n}\right),$$

$$\frac{1}{n}\langle \mathrm{Tr}S^8\rangle=14\langle\xi^2\rangle^4+O\left(\frac{1}{n}\right), \tag{3.2}$$

which are easily seen as the lowest moments of the semicircle density[3]

$$\frac{1}{n}\langle \mathrm{Tr}S^{2k}\rangle=C_k\langle\xi^2\rangle^k+O\left(\frac{1}{n}\right),$$

$$C_k=\frac{1}{k+1}\binom{2k}{k}=\frac{1}{2\pi}\int_{-2}^{2}dx\sqrt{4-x^2}x^{2k}. \tag{3.3}$$

The variance $\sigma^2=\langle\xi^2\rangle$ of the probability density of the $\xi_{i,j}$ random variables is the only parameter entering in the limiting spectral density

$$G(z)=\lim_{n\to\infty}\frac{1}{n}\mathrm{E}\left[\mathrm{Tr}\frac{1}{z-S}\right]$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{p=0}^{\infty}\frac{1}{z^{p+1}}\mathrm{E}[\mathrm{Tr}S^p]=\sum_{k=0}^{\infty}\frac{1}{z^{2k+1}}C_k\sigma^{2k}$$
$$=\frac{z}{2\pi}\int_{-2}^{2}dx\frac{\sqrt{4-x^2}}{z^2-x^2\sigma^2}=\frac{1}{2\sigma^2}(z-\sqrt{z^2-4\sigma^2}),$$

$$\rho(\lambda)=-\frac{1}{\pi}\,\mathrm{Im}\,G(\lambda+i\epsilon)=\begin{cases}\dfrac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-\lambda^2} & \text{if } |\lambda|<2\sigma\\ 0 & \text{if } |\lambda|>2\sigma\end{cases}. \tag{3.4}$$

B. Random graph scaling

The simplest model of random graphs is the ensemble $\{S_0\}$, with the probability density

$$s_{i,j}=\begin{cases}1, & \text{with probability } 1/n,\\ 0, & \text{with probability } 1-1/n.\end{cases} \tag{3.5}$$

Then $E[(s_{i,j})^k]=\langle v^k\rangle=1/n$ for every k. By inserting this into Eqs. (2.10) and (2.12) and by keeping only the leading-order contribution in the limit $n\to\infty$, one finds that all the odd moments and, therefore, the odd powers of traces are negligible. The limiting form of the equations become

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr}S^2\rangle=1,$$

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr}S^4\rangle=3,$$

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr}S^6\rangle=12,$$

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr}S^8\rangle=57. \tag{3.6}$$

These numbers reproduce the asymptotic behavior in the limit $n\to\infty$ of the moments $E[(1/n)\mathrm{Tr}\,S^{2p}]=m_p$, evaluated by Khorunzhy and Vengerovsky [3]. They also found a recursive relation that evaluates the asymptotic moments m_p and studied their behavior at increasing values of p. The same recursive relation was obtained and analyzed by Bauer and Golinelli [1].

A very similar scaling is used in models of sparse matrices

$$\langle (s_{i,j})^k\rangle=\langle v^k\rangle=\frac{1}{n}c_k. \tag{3.7}$$

It is easy to see from Eqs. (2.10) and (2.12) that with this rescaling the same limit holds for matrices of the two ensembles $\{S\}$ and $\{S_0\}$ and that the odd moments vanish

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr}S^{2p+1}\rangle=0,$$

[3]It may be useful to recall that the semicircle density was obtained as the limiting spectral density for ensembles of real matrices more general than the ensembles $\{S_0\}$ and $\{S\}$ discussed in this paper. Indeed it is not necessary that the independent matrix entries have the same probability distribution nor is it necessary that all the moments of the entries should exist (see, for instance, [23]).

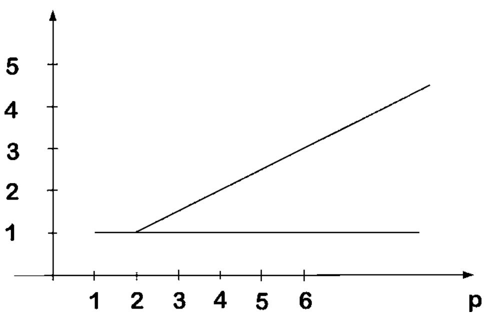

FIG. 10. The rescaling power $a(p)$. The line $a=1$ is the rescaling used in random graphs, the line $a(p)=p/2$ is the Wigner scaling.

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr} S^2\rangle = c_2,$$

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr} S^4\rangle = c_4 + 2(c_2)^2,$$

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr} S^6\rangle = c_6 + 6c_4c_2 + 5(c_2)^3,$$

$$\lim_{n\to\infty}\frac{1}{n}\langle \mathrm{Tr} S^8\rangle = c_8 + 8c_6c_2 + 6(c_4)^2 + 28c_4(c_2)^2 + 14(c_2)^4. \tag{3.8}$$

C. Intermediate scalings

In both above-mentioned scalings, the Wigner scaling and the sparse matrix scaling, the second moment of the entries is scaled with the same power

$$\langle (s_{i,j})^2\rangle = \langle v^2\rangle = \frac{1}{n}c_2.$$

Every moment $\langle v^k\rangle$ with $k\geqslant 2$, is scaled by the sparse matrix scaling (3.7) in a way sufficient to obtain finite asymptotic limits $\lim_{n\to\infty}(1/n)\langle \mathrm{Tr}\ S^p\rangle$. Furthermore, the same scaling suppresses asymptotically every contribution of the first moment $\langle v\rangle = c_1/n$.

If the first moment $\langle v\rangle$ is not zero, and the matrix entries have Wigner scaling, $\langle v^k\rangle = c_k/n^{k/2}$ [see Eq. (3.1)], several rescaled $E[\mathrm{Tr}\ S^p]$ or $E[\mathrm{Tr}\ S_0^p]$ diverge in the $n\to\infty$ limit. We considered a simple Gaussian example in the Appendix. Most authors assume the first moment $\langle v\rangle=0$ together with Wigner scaling, as done at the beginning of this section. Wigner scaling asymptotically suppresses every contribution $\langle v^k\rangle$ with $k>2$ and in this way obtains universality.

Let us parametrize possible scalings of the moments of the probability distribution of the entries

$$\langle (s_{i,j})^k\rangle = \langle v^k\rangle = \frac{1}{n^{a(k)}}c_k. \tag{3.9}$$

Figure 10 plots the rescaling power $a(p)$ on the vertical axes. The horizontal line $a(p)=1$ corresponds to the sparse matrix scaling; the line $a(p)=p/2$, with $p\geqslant 2$ corresponds to Wigner scaling.

Any line in the sector between the two lines, that is, any scaling with $a(p)=\alpha p + 1 - 2\alpha$ with $0<\alpha<1/2$ and $p\geqslant 2$, supplemented by $\langle v\rangle=0$, would obtain finite asymptotic evaluations for $(1/n)\langle \mathrm{Tr}\ S^p\rangle$. Our Eqs. (2.10) and (2.12), up to $p=8$ suggest that *such asymptotic evaluations would be identical with Wigner scaling*, provided the first moment vanishes, $\langle v\rangle=0$.

A simple example of distribution probability corresponding to intermediate scaling is

$$s_{i,j} = \begin{cases} 0, & \text{with probability } \quad 1-\dfrac{1}{n^{1-2\alpha}}, \\ -\dfrac{1}{n^{\alpha}}, & \text{with probability } \quad \dfrac{1}{2n^{1-2\alpha}}, \\ \dfrac{1}{n^{\alpha}}, & \text{with probability } \quad \dfrac{1}{2n^{1-2\alpha}}, \end{cases} \tag{3.10}$$

where $0<\alpha<1/2$; then, one evaluates

$$\langle (s_{i,j})^k\rangle = \begin{cases} 0, & \text{if } k \text{ odd}, \\ \dfrac{1}{n^{\alpha(k-2)+1}}, & \text{if } k \text{ even}. \end{cases} \tag{3.11}$$

APPENDIX

The effects of a nonvanishing expectation value $\langle v\rangle$ may be seen in the simple model of the ensemble $\{S\}$ of real symmetric matrices with normal probability distribution for the entries. The joint probability density factorizes

$$p(s_{1,2},s_{1,3},\ldots,s_{n-1,n}) = \prod_{i<j} p(s_{i,j}), \tag{A1}$$

$$p(s_{i,j}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(s_{ij}-v)^2/2\sigma^2}.$$

The first moments $\langle v^k\rangle$ are well known

$$\langle s_{ij}\rangle = v, \quad \langle (s_{ij})^2\rangle = \sigma^2 + v^2,$$

$$\langle (s_{ij})^3\rangle = 3\sigma^2 v + v^3, \quad \langle (s_{ij})^4\rangle = 3\sigma^4 + 6\sigma^2 v^2 + v^4,$$

$$\langle (s_{ij})^5\rangle = 15\sigma^4 v + 10\sigma^2 v^3 + v^5, \quad \langle (s_{ij})^6\rangle = 15\sigma^6 + 45\sigma^4 v^2 + 15\sigma^2 v^4 + v^6,$$

$$\langle (s_{ij})^7\rangle = 105\sigma^6 v + 105\sigma^4 v^3 + 21\sigma^2 v^5 + v^7. \tag{A2}$$

By inserting these moments, after Wigner rescaling $s_{ij}\to s_{ij}/\sqrt{n}$, in Eq. (2.12), one finds the large n behavior of $\langle \mathrm{Tr}\ S^p\rangle$ for $p=1,2,\ldots,7$

$$\frac{1}{n}\langle \mathrm{Tr} S\rangle = v/\sqrt{n},$$

$$\frac{1}{n}\langle \mathrm{Tr} S^2\rangle = (\sigma^2 + v^2),$$

$$\frac{1}{n}\langle \mathrm{Tr}S^3\rangle = n^{1/2}v^3 + 3n^{-1/2}\sigma^2 v,$$

$$\frac{1}{n}\langle \mathrm{Tr}S^4\rangle = nv^4 + 2(\sigma^2+2v^2)\sigma^2 + \frac{1}{n}(\sigma^2+4v^2)\sigma^2 + O\left(\frac{1}{n^2}\right),$$

$$\frac{1}{n}\langle \mathrm{Tr}S^5\rangle = n^{3/2}v^5 + 5n^{1/2}\sigma^2 v^3 + \frac{10}{n^{1/2}}(\sigma^2+v^2)\sigma^2 v + O\left(\frac{1}{n^{3/2}}\right),$$

$$\frac{1}{n}\langle \mathrm{Tr}S^6\rangle = n^2v^6 + 6n\sigma^2 v^4 + (11\sigma^4+15\sigma^2v^2+18v^4)\sigma^2 + O\left(\frac{1}{n}\right),$$

$$\frac{1}{n}\langle \mathrm{Tr}S^7\rangle = n^{5/2}v^7 + 7n^{3/2}\sigma^2 v^5 + 7n^{1/2}(3\sigma^2+4v^2)\sigma^2 v^3 + O(1). \tag{A3}$$

The large-n resolvent $G(z)$ of this ensemble may be investigated with the addition law of random matrices [24–27].

Any real symmetric matrix of the ensemble $\{S\}$, where $\langle v\rangle$ is not equal to zero, is the sum of a random matrix of the same ensemble, with $\langle v\rangle=0$ and the fixed matrix vJ, where the J matrix has all entries equal to one. After Wigner scaling, all the entries of the random and fixed matrices have the further scaling factor $n^{-1/2}$.

The resolvent $G(z)$ and its inverse function, for the fixed matrix $vJn^{-1/2}$ are

$$G(z) = \frac{n-1}{n}\frac{1}{z} + \frac{1}{n}\frac{1}{z-v\sqrt{n}},$$

$$z = \frac{1+\sqrt{n}vG+\sqrt{(1-\sqrt{n}vG)^2+4n^{-1/2}vG}}{2G}. \tag{A4}$$

The large-n limit for the resolvent and its inverse function, for the ensemble of Gaussian matrices with $\langle v\rangle=0$, correspond to the Wigner semicircle

$$G(z) = \frac{1}{2\sigma^2}(z-\sqrt{z^2-4\sigma^2}),$$

$$z = \frac{1}{G} + \sigma^2 G. \tag{A5}$$

According to the law of addition of random matrices, the resolvent and its inverse function, for the ensemble $\{Sn^{-1/2}\}$, for large-n have the Pastur form

$$G(z) = \frac{1}{n}\sum_j \frac{1}{z-\epsilon_j-\sigma^2 G} = \frac{n-1}{n}\frac{1}{z-\sigma^2 G} + \frac{1}{n}\frac{1}{z-v\sqrt{n}-\sigma^2 G},$$

$$z = \sigma^2 G + \frac{1+\sqrt{n}vG+\sqrt{(1-\sqrt{n}vG)^2+4n^{-1/2}vG}}{2G}. \tag{A6}$$

On the right-hand side of Eq. (A6) for the resolvent $G(z)$, the first term dominates the second one for every z finite and n large, whereas the second term is dominant for z very close to $v\sqrt{n}$. Then a simple approximate form follows by considering only one term:

$$G(z) \sim \begin{cases} \dfrac{1}{2\sigma^2}(z-\sqrt{z^2-4(1-1/n)\sigma^2}) \\ \quad z \text{ finite and } n \text{ large}, \\ \dfrac{1}{2\sigma^2}(z-v\sqrt{n}-\sqrt{(z-v\sqrt{n})^2-4\sigma^2/n}) \\ \quad z \sim v\sqrt{n}+\dfrac{\sigma^2}{v\sqrt{n}}, \end{cases}$$

corresponding to the spectral densities

$$\rho(\lambda) \sim \frac{1}{2\pi\sigma^2}\sqrt{4(1-1/n)\sigma^2-\lambda^2}, \quad \text{if } |\lambda| < 2\sigma\sqrt{(1-1/n)}, \quad n \text{ large},$$

$$\rho(\lambda) \sim \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2/n-(\lambda-v\sqrt{n})^2}, \quad \text{if } |\lambda-v\sqrt{n}| < 2\sigma/\sqrt{n}.$$

Finally, Eq. (A6) for $z=z(G)$ is easily expanded for small values of G, the expansion is inverted obtaining the large z expansion of the resolvent, as it follows from the addition law:

$$G(z) = \frac{1}{z} + \frac{v}{n^{1/2}}\frac{1}{z^2} + \frac{v^2+\sigma^2}{z^3} + \left(n^{1/2}v^2+\frac{3}{n^{1/2}}\sigma^2\right)v\frac{1}{z^4} + \left(nv^4 + 2(\sigma^2+2v^2)\sigma^2 + \frac{2}{n}\sigma^2v^2\right)\frac{1}{z^5} + (n^{3/2}v^5+5n^{1/2}v^3\sigma^2 + 5n^{-1/2}v\sigma^2(v^2+2\sigma^2))\frac{1}{z^6} + \left(n^2v^6+6n\sigma^2v^4+(5\sigma^4 + 15\sigma^2v^2+9v^4)\sigma^2+\frac{15}{n}\sigma^4v^2\right)\frac{1}{z^7} + (n^{7/2}v^7+7n^{3/2}\sigma^2v^5 + 7n^{1/2}(2v^2+3\sigma^2)\sigma^2v^3)\frac{1}{z^8} + \cdots . \tag{A7}$$

By comparing this expansion with the direct evaluation of the large n behavior of $E(\mathrm{Tr}\ S^p)$ given in (A3), we see that the first two terms of the expansion are correctly reproduced for every $p=1,2,\ldots,7$. Analogous analysis and conclusions are abtained for the $\{S_0\}$ ensemble.

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